

Dirac-Born-Infeld Field Trapped in the Braneworld

Ricardo García-Salcedo,^{1, a} Dania Gonzalez,^{2, b} Tame Gonzalez,^{3, c} Claudia Moreno,^{4, d} and Israel Quiros^{3, 5, e}

¹*Centro de Investigacion en Ciencia Aplicada y Tecnologia Avanzada - Legaria del IPN, México D.F., México.*

²*Departamento de Matemática, Universidad Central de Las Villas, 54830 Santa Clara, Cuba.*

³*Departamento de Física, Universidad Central de Las Villas, 54830 Santa Clara, Cuba.*

⁴*Departamento de Física y Matemáticas, Centro Universitario de Ciencias Exáctas e Ingenierías, Corregidora 500 S.R., Universidad de Guadalajara, 44420 Guadalajara, Jalisco, México.*

⁵*Instituto de Física de la Universidad de Guanajuato, A.P. 150, 37150, León, Guanajuato, México.*

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We apply the dynamical systems tools to study the (linear) cosmic dynamics of a Dirac-Born-Infeld-type field trapped in the braneworld. We will focus, exclusively, in Randall-Sundrum and Dvali-Gabadadze-Porrati brane models. We analyze the existence and stability of asymptotic solutions for the AdS throat and the quadratic potential. It is demonstrated, in particular, that the ultra-relativistic approximation matter-scaling and scalar field-dominated solutions always arise. In the first scenario the empty universe is the past attractor, while in the second model the past attractor is matter or kinetic dominated phase.

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I. INTRODUCTION

Recent observations from the Wilkinson Microwave Anisotropy Probe (WMAP) [1] offer strong supporting evidence in favor of the inflationary paradigm. In the most simple models of this kind, the energy density of the universe is dominated by the potential energy of a single (inflaton) scalar field that slowly rolls down in its self-interaction potential [2]. Restrictions imposed upon the class of potentials which can lead to realistic inflationary scenarios, are dictated by the slow-roll approximation, and hence, the result is that only sufficiently flat potentials can drive inflation. In order for the potential to be sufficiently flat, these conventional inflationary models should be fine-tuned. This simple picture of the early-time cosmic evolution can be drastically changed if one considers models of inflation inspired in “Unified Theories” like the Super String or M-theory. The most appealing models of this kind are the Randall-Sundrum braneworld model of type 2 (RS2) [3] and Dvali-Gabadadze-Porrati (DGP) brane worlds [4].

In the RS2 model a single codimension 1 brane with positive tension is embedded in a five-dimensional anti-de Sitter (AdS) space-time, which is infinite in the direction perpendicular to the brane. In general, the standard model particles are confined to the brane, meanwhile

gravitation can propagate in the bulk. In the low-energy limit, due to the curvature of the bulk, the graviton is confined to the brane, and standard (four-dimensional) general relativity laws are recovered. RS2 braneworld models have an appreciable impact on early universe cosmology, in particular, for the inflationary paradigm. In effect, a distinctive feature of cosmology with a scalar field confined to a RS2 brane is that the expansion rate of the universe differs at high energy from that predicted by standard general relativity. This is due to a term – quadratic in the energy density – that produces enhancing of the friction acting on the scalar field. This means that, in RS2 braneworld cosmology, inflation is possible for a wider class of potentials than in standard cosmology [5]. Even potentials that are not sufficiently flat from the point of view of the conventional inflationary paradigm can produce successful inflation. At sufficiently low energies (much less than the brane tension), the standard cosmic behavior is recovered prior to primordial nucleosynthesis scale ($T \sim 1 \text{ MeV}$) and a natural exit from inflation ensues as the field accelerates down its potential [6].¹

The DGP model describes a brane with 4D world-volume, that is embedded into a flat 5D bulk, and allows for infrared (IR)/large scale modifications of grav-

^aElectronic address: rigarcias@ipn.mx

^bElectronic address: dgm@uclv.edu.cu

^cElectronic address: tame@uclv.edu.cu

^dElectronic address: claudia.moreno@ucei.udg.mx

^eElectronic address: israel@uclv.edu.cu

¹ In this scenario, reheating arises naturally even for potentials without a global minimum and radiation is created through gravitational particle production [7] and/or through curvaton reheating [8]. This last ingredient improves the brane “steep” inflationary picture [9]. Other mechanisms such as preheating, for instance, have also been explored [10].

itational laws. A distinctive ingredient of the model is the induced Einstein-Hilbert action on the brane, that is responsible for the recovery of 4D Einstein gravity at moderate scales, even if the mechanism of this recovery is rather non-trivial [11]. Nevertheless, studying the dynamics of DGP models continues being a very attractive subject of research [12]. It is due, in part, to the very simple geometrical explanation to the “dark energy” problem and the fact that it is one of a very few possible consistent IR modifications of gravity that might be ever found. The acceleration of the expansion at late times is explained here as a consequence of the leakage of gravity into the bulk at large (cosmological) scales, which is just a 5D geometrical effect.

Nonlinear scalar field theories of the Dirac-Born-Infeld (DBI) type [13, 14, 15] have attracted much attention in recent years due to their role in models of inflation based on string theory [16]. These scenarios identify the inflaton with the position of a mobile D-brane moving on a compact 6-dimensional submanifold of spacetime (for reviews and references see [17]), which means that the inflaton is interpreted as an open string mode. Usually only effective four-dimensional DBI cosmological models are studied.

In our opinion that studying the impact higher-dimensional brane effects has on the cosmic dynamics of DBI-type models, is a task worthy of attention. Aim of this paper is, precisely, to investigate the dynamics of a DBI-type field trapped in a Randall-Sundrum brane and in a Dvali-Gabadadze-Porrati braneworld, respectively, by invoking the dynamical systems tools. The study of the asymptotic properties of these models allows to correlate such important dynamical systems concepts like past and future attractors – as well as saddle equilibrium points – with generic cosmological evolution. A DBI field trapped in a RS brane could be a nice scenario to make early-time inflation easier, meanwhile a DBI scalar field confined to a (self-accelerating) DGP braneworld could be a useful arena where to address unified description of early-time inflation and late-time speed-up.

In a sense the present work might be considered as a natural completion of the one reported in Ref. [14]. For this reason, as in the above reference, here we concentrate in the case of an anti-de Sitter (AdS) throat and quadratic self-interaction potential for the inflaton. In addition to the scalar field we also consider a background fluid trapped on the braneworld. Through the paper we use natural units ($8\pi G = 8\pi/m_{Pl}^2 = \hbar = c = 1$).

II. DBI ACTION

Consider the following effective action for a D3-brane:

$$S_{DBI} = - \int \frac{d^4x}{g_{YM}^2} \sqrt{|g|} \{ f^{-1}(\phi) \sqrt{1 + f(\phi)(\nabla\phi)^2} - f^{-1}(\phi) + V(\phi) \}, \quad (1)$$

where g_{YM}^2 is the Yang-Mills coupling and $V(\phi)$ is the potential for the DBI-type field. For a spatially flat FRW metric $(\nabla\phi)^2 = -\dot{\phi}^2$, where the dot accounts for derivative in respect to the cosmic time. The equation of motion for the DBI-type scalar field ϕ can be written in the following way:

$$\ddot{\phi} + \frac{3\partial_\phi f}{2f} \dot{\phi}^2 - \frac{\partial_\phi f}{f^2} + 3\gamma_L^{-2} H \dot{\phi} + \gamma_L^{-3} \left(\partial_\phi V + \frac{\partial_\phi f}{f^2} \right) = 0, \quad (2)$$

where the Lorentz boost (factor) γ_L is defined as

$$\gamma_L = \frac{1}{\sqrt{1 - f(\phi)\dot{\phi}^2}}. \quad (3)$$

Alternatively the motion equation of the DBI-type field can be written in the form of a continuity equation:

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0, \quad (4)$$

where we have defined the following energy density and pressure of the DBI scalar field:

$$\begin{aligned} \rho_\phi &= \frac{\gamma_L - 1}{f} + V(\phi), \\ p_\phi &= \frac{\gamma_L - 1}{\gamma_L f} - V(\phi), \end{aligned} \quad (5)$$

respectively.

In this paper we concentrate just on a AdS throat whose amounts to consider $f(\phi) = \alpha/\phi^4$, where α is the 't Hooft coupling, which is related to g_{YM}^2 – the Yang-Mills coupling – via the relation $\alpha = g_{YM}^2 N$ in the large- N limit of the field theory, and a quadratic self-interaction potential $V(\phi) = m^2 \phi^2/2$.

III. AUTONOMOUS SYSTEMS

The dynamical systems tools offer a very useful approach to the study of the asymptotic properties of the cosmological models [18]. In order to be able to apply these tools one has to (unavoidably) follow the steps enumerated below.

- First: to identify the phase space variables that allow writing the system of cosmological equations in the form of an autonomous system of ordinary differential equations (ODE). There can be several different possible choices, however, not all of them allow for the minimum possible dimensionality of the phase space.
- Next: with the help of the chosen phase space variables, to build an autonomous system of ODE out of the original system of cosmological equations.
- Finally, a some times forgotten or under-appreciated step: to identify the phase space spanned by the chosen variables, that is relevant to the cosmological model under study.

After this one is ready to apply the standard tools of the (linear) dynamical systems analysis.

The goal of the dynamical systems study is to correlate such important concepts like past and future attractors (also, saddle critical points) with asymptotic cosmological solutions. If a given cosmological solution can be associated with a critical point in the phase space of the model, this means that, the universe's dynamics will evolve for a sufficiently long time in the neighbourhood of this solution, otherwise, it represents a quite generic phase of the cosmic dynamics.

In the following subsections we keep the expressions as general as possible, and then, in section IV we substitute the above mentioned expressions for $f(\phi)$ and $V(\phi)$.

A. The DBI-RS Model

Here we will be concerned with the dynamics of a Dirac-Born-Infeld field that is trapped in a Randall-Sundrum brane of type 2. The field equations – in terms of the Friedmann-Robertson-Walker (FRW) metric – are the following:

$$3H^2 = \rho_T \left(1 + \frac{\rho_T}{2\lambda}\right), \quad (6)$$

$$2\dot{H} = -(1 + \frac{\rho_T}{\lambda}) \left[\gamma_L \dot{\phi}^2 + (1 + \omega_m) \rho_m \right], \quad (7)$$

where ω_m is the equation of state (EOS) parameter of the matter fluid, while $\rho_T = \rho_\phi + \rho_m$ – the total energy density on the brane. Additionally one has to consider the continuity equations for the DBI-type field (equation (2)) or, alternatively, (4)) and for the matter fluid:

$$\dot{\rho}_m + 3(1 + \omega_m)H\rho_m = 0. \quad (8)$$

The model described by the above equations will be called as “DBI-RS model”.

Our aim will be to write the latter system of second-order (partial) differential equations, as an autonomous system of (first order) ordinary differential equations. For this purpose we introduce the following phase variables [14]:

$$\begin{aligned} x &\equiv \frac{1}{H} \sqrt{\frac{\gamma_L}{3f}}, \quad y \equiv \frac{\dot{\phi} \sqrt{\gamma_L}}{H}, \quad z \equiv \frac{\sqrt{V}}{\sqrt{3}H}, \quad r \equiv \frac{\rho_T}{3H^2}, \\ \mu_1 &\equiv \frac{\partial_\phi V}{V^{3/2} f^{1/2}}, \quad \mu_2 \equiv \frac{\partial_\phi f}{V^{3/2} f^{5/2}}. \end{aligned} \quad (9)$$

It can be realized that, in terms of the variable r ,

$$\frac{\rho_T}{\lambda} = \frac{2(1-r)}{r}, \Rightarrow 0 < r \leq 1. \quad (10)$$

This means that the four-dimensional (low-energy) limit of the Randall-Sundrum cosmological equations – corresponding to the formal limit $\lambda \rightarrow \infty$ – is associated with the value $r = 1$. The high-energy limit $\lambda \rightarrow 0$, on the contrary, corresponds to $r \rightarrow 0$.

We write the Lorentz boost in terms of the variables of phase space as:

$$\gamma \equiv \frac{1}{\gamma_L} = \sqrt{1 - \frac{y^2}{3x^2}}. \quad (11)$$

Standard “non-relativistic” behaviour corresponds to $\gamma = 1$, while the “ultra-relativistic” (UR) regime is associated with $\gamma = 0$.

In terms of the variables that span the phase space $(x, y, z, r, \mu_1, \mu_2)$ the cosmological equations (4), (6), (7), (8) can be written as an autonomous system of ordinary differential equations (ODE):

$$x' = -\frac{1}{2}(\mu_1 + \mu_2) \frac{yz^3}{x^2} - \frac{y^2}{2x} - x \frac{H'}{H}, \quad (12)$$

$$\begin{aligned} y' &= -\frac{3}{2}[\mu_1(\gamma^2 + 1) + \mu_2(\gamma - 1)^2] \frac{z^3}{x} \\ &\quad - \frac{3}{2}(\gamma^2 + 1)y - y \frac{H'}{H}, \end{aligned} \quad (13)$$

$$z' = \frac{1}{2}\mu_1 \frac{yz^2}{x} - z \frac{H'}{H}, \quad (14)$$

$$r' = \frac{2r(r-1)}{2-r} \frac{H'}{H}, \quad (15)$$

$$\mu_1' = \mu_1^2 \frac{yz}{x} \left[\Gamma_V - \frac{3}{2} - \frac{\partial_\phi \ln f}{\partial_\phi \ln V^2} \right], \quad (16)$$

$$\mu_2' = \frac{\mu_2^2}{\gamma} \frac{yz^3}{x^3} \left[\Gamma_f - \frac{5}{2} - \frac{\partial_\phi \ln V^3}{\partial_\phi \ln f^2} \right], \quad (17)$$

where the prime denotes derivative with respect to the number of e-foldings $\tau \equiv \ln a$, while $\Gamma_V \equiv (V\partial_\phi^2 V)/(\partial_\phi V)^2$, $\Gamma_f \equiv (f\partial_\phi^2 f)/(\partial_\phi f)^2$ and:

$$\frac{H'}{H} = -\frac{2-r}{2r} \{y^2 + 3(\omega_m + 1)[r - (1-\gamma)x^2 - z^2]\}. \quad (18)$$

It will be helpful to have the parameters of observational importance $\Omega_\phi = \rho_\phi/3H^2$ – the scalar field dimensionless energy density parameter, and the equation of state (EOS) parameter $\omega_\phi = p_\phi/\rho_\phi$, written in terms of the variables of phase space:

$$\Omega_\phi = (1-\gamma)x^2 + z^2, \quad \omega_\phi = \frac{(1-\gamma)\gamma x^2 - z^2}{(1-\gamma)x^2 + z^2}. \quad (19)$$

Recall, also, that the deceleration parameter is $q = -(1 + H'/H)$:

$$q = -1 + \frac{2-r}{2r} \{y^2 + 3(\omega_m + 1)[r - (1-\gamma)x^2 - z^2]\}.$$

B. DBI-DGP model

In this section we focus our attention in a braneworld model where a DBI-type field is confined to a DGP brane. In the (flat) FRW metric, the cosmological (field) equations are the following:

$$\begin{aligned} Q_\pm^2 &= \frac{1}{3}(\rho_m + \rho_\phi), \\ \dot{\rho}_m &= -3(1 + \omega_m)H\rho_m, \end{aligned} \quad (20)$$

where, as before ω_m is the EOS parameter of the matter fluid, ρ_m is the energy density of the background barotropic fluid and ρ_ϕ is the energy density of DBI field. Also one has to consider the continuity equations for the DBI-type field (equation (2) or, alternatively, equation (4)). We have used the following definition:

$$Q_\pm^2 \equiv H^2 \pm \frac{1}{r_c} H, \quad (21)$$

with r_c being the so called crossover scale. In what follows we will refer to this model as the “DBI-DGP model”.

There are two possible branches of the DGP model corresponding to the two possible choices of the signs in (21): “+” is for the normal DGP models that are free of ghost, while “-” is for the self-accelerating solution.

As before, our goal will be to write the latter system of second-order (partial) differential equations, as an autonomous system of (first order) ordinary differential

equations. For this purpose we introduce the following phase variables:

$$x \equiv \frac{1}{Q} \sqrt{\frac{\gamma_L}{3f}}, \quad y \equiv \frac{\dot{\phi} \sqrt{\gamma_L}}{Q}, \quad z \equiv \frac{\sqrt{V}}{\sqrt{3}Q}, \quad r \equiv \frac{Q}{H}. \quad (22)$$

The expression determining the Lorentz boost coincides with (11):

$$\gamma \equiv \frac{1}{\gamma_L} = \sqrt{1 - \frac{y^2}{3x^2}}.$$

Hence, as before, $\gamma = 1$ is for the non-relativistic case, while $\gamma = 0$ is for the UR regime.

After the above choice of phase space variables the cosmological equations can be written as an autonomous system of ODE:

$$x' = -\frac{y}{2x^2} [xy + z^3 r(\mu_1 + \mu_2)] - x \frac{Q'}{Q}. \quad (23)$$

$$\begin{aligned} y' &= -\frac{3z^3 r}{2x} [\mu_1(\gamma^2 + 1) + \mu_2(\gamma - 1)^2] - \\ &\quad - \frac{3y}{2}(\gamma^2 + 1) - y \frac{Q'}{Q}, \end{aligned} \quad (24)$$

$$z' = \frac{yz^2 r}{2x} \mu_1 - z \frac{Q'}{Q}, \quad (25)$$

$$r' = r \left(\frac{1-r^2}{1+r^2} \right) \frac{Q'}{Q}. \quad (26)$$

Recall that the prime denotes derivative with respect to the number of e-foldings $\tau \equiv \ln a$, while

$$\mu_1 \equiv \frac{\partial_\phi V}{V^{3/2} f^{1/2}}, \quad \mu_2 \equiv \frac{\partial_\phi f}{V^{3/2} f^{5/2}}.$$

We have considered also the following relationship:

$$\frac{H'}{H} = \frac{2r^2}{1+r^2} \frac{Q'}{Q},$$

where

$$\frac{Q'}{Q} \equiv -\frac{1}{2} \{3(1 + \omega_m)[1 - x^2(1 - \gamma) - z^2] + y^2\}. \quad (27)$$

Equations (23)-(26) have to be complemented with the addition of equations (16) and (17) above, which are the autonomous ordinary differential equations for μ_1 and μ_2 , respectively.

Equation (21) can be rewritten as:

$$r^2 = 1 \pm \frac{1}{r_c H}. \quad (28)$$

For the Minkowski phase, since $0 \leq H \leq \infty$ (we consider just non-contracting universes), then $1 \leq r \leq \infty$. The case $-\infty \leq r \leq -1$ corresponds to the time reversal of the latter situation. For the self-accelerating phase, $-\infty \leq r \leq 1$, but since we want real valued r only, then $0 \leq r \leq 1$.² As before, the case $-1 \leq r \leq 0$ represents time reversal of the case $0 \leq r \leq 1$ that will be investigated here.³ Both branches share the common subset $(x, y, z, r = 1)$, which corresponds to the formal limit $r_c \rightarrow \infty$ (see equation (21)), i.e., this represents just the standard behaviour typical of Einstein-Hilbert theory coupled to a self-interacting scalar field.

To finalize this section we write useful magnitudes of observational interest in terms of the dynamical variables (22). The “effective” dimensionless density parameters $\bar{\Omega}_m = \rho_m/3Q^2$ and $\bar{\Omega}_\phi = \rho_\phi/3Q^2$:

$$\begin{aligned}\bar{\Omega}_m &= \frac{\Omega_m}{r^2} = 1 - x^2(1 - \gamma) - z^2, \\ \bar{\Omega}_\phi &= \frac{\Omega_\phi}{r^2} = x^2(1 - \gamma) + z^2.\end{aligned}\quad (29)$$

For the equation of state (EOS) $\omega_\phi = p_\phi/\rho_\phi$:

$$\omega_\phi = \frac{x^2(1 - \gamma)\gamma - z^2}{x^2(1 - \gamma) + z^2}.\quad (30)$$

C. Brane Effects in the Phase Space

Higher-dimensional (brane) effects are encrypted in the variable r for the DBI-RS model as well as for the DBI-DGP model. In both cases the hyper-surfaces in the phase space that are foliated by the value $r = 1$: $Hyp = (x, y, z, r = 1, \mu_1, \mu_2)$, represent the loci of the equilibrium configurations that can be associated with standard general relativity behaviour. For the DBI-DGP model the hyper-plane Hyp represents, additionally, the boundary in the phase space separating the Minkowski cosmological phase ($r \geq 1$) from the self-accelerating one ($0 < r \leq 1$).

Trajectories in the phase space that depart from the phase hyper-plane Hyp are associated with higher-dimensional modifications of general relativity produced by the brane effects.

IV. EQUILIBRIUM POINTS IN THE PHASE SPACE

In this section we will analyze in detail the existence and stability of critical points of the autonomous systems corresponding to both Randall-Sundrum and Dvali-Gabadadze-Porrari braneworld models. In both cases we study an AdS throat – often explored in the literature – and quadratic potential: $f(\phi) = \alpha/\phi^4$, and $V(\phi) = m^2\phi^2/2$ [14], that amounts to the following relationship between f and V :

$$f = \frac{\alpha m^4}{4V^2}.$$

The above choice leads to $\mu_1 = \mu$ being a constant

$$\mu = \sqrt{2/\alpha}(2/m),$$

while

$$\mu_2 = -2\gamma(x^2/z^2)\mu.$$

Hence, of the three variables x , z , and μ_2 , only two (say x and z) are independent. This fact leads to considerable simplification of the problem since the dimension of the autonomous system reduces from six down to four. This is one of the reasons why the present particular case is generously considered in the bibliography.

A. DBI-RS Model

As just noticed, after considering the specific form of the functions $f(\phi)$ and $V(\phi)$ above, the six-dimensional autonomous dynamical system (12-17) can be reduced down to a four-dimensional one:

$$\begin{aligned}x' &= -\frac{\mu y z}{2x^2}(z^2 - 2\gamma x^2) - \frac{y^2}{2x} - x\frac{H'}{H}, \\ y' &= 3\mu\gamma(1 - \gamma)^2 x z - \frac{3\mu(1 + \gamma^2)z^3}{2x} \\ &\quad - \frac{3(1 + \gamma^2)y}{2} - y\frac{H'}{H}, \\ z' &= \frac{\mu y z^2}{2x} - z\frac{H'}{H}, \quad r' = \frac{2r(r - 1)}{2 - r}\frac{H'}{H},\end{aligned}\quad (31)$$

where the ratio $H'/H = -(q + 1)$ is given by Eq. (18). It arises the following constraint:

$$\Omega_m = \frac{\rho_m}{3H^2} = r - (1 - \gamma)x^2 - z^2.\quad (32)$$

For $\Omega_m \geq 0$, then $(1 - \gamma)x^2 + z^2 \leq r$. Besides, for $\Omega_m \leq 1$ then $(1 - \gamma)x^2 + z^2 \geq r - 1$. We will be focused

² In fact, fitting SN observations requires $H \geq r_c^{-1}$ in order to achieve late-time acceleration (see, for instance, reference [19] and references therein). This means that r has to be real-valued.

³ Points with $r = 0$ and their neighbourhood have to be carefully analyzed due to the fact that at $r = 0$, other phase space variables (see Eq.(22)) and equations in (23)-(26) are undefined.

TABLE I: Properties of the critical points for the autonomous system (31).

Equilibrium point	x	y	z	r	Existence	γ	Ω_m	Ω_ϕ	ω_ϕ	q
E	0	0	0	0	Always	Undefined	0	0	Undefined	$3\omega_m + 2$
M	0	0	0	1	"	Undefined	1	0	Undefined	$(3\omega_m + 1)/2$
U^\pm	1	$\pm\sqrt{3}$	0	1	"	0	0	1	0	$1/2$

TABLE II: Eigenvalues of the linearization matrices corresponding to the critical points in table I.

Equilibrium point	λ_1	λ_2	λ_3	λ_4
E	$3\omega_m + 2$	$3(\omega_m + 1)$	$3(\omega_m + 1)$	$3(\omega_m + 1)$
M	$-3(1 + \omega_m)$	$3(\omega_m - 1)/2$	$3(\omega_m + 1)/2$	$(3\omega_m + 1)/2$
U^\pm	$-3\omega_m$	-3	$3/2$	3

on expanding FRW universes, so that $x \geq 0$ and $z \geq 0$. The resulting four-dimensional phase space for the DBI-RS model is the following:

$$\Psi = \{(x, y, z, r) : r - 1 \leq (1 - \gamma)x^2 + z^2 \leq r, \\ x \geq 0, y^2 \leq 3x^2, z \geq 0, 0 < r \leq 1\}. \quad (33)$$

As properly noticed in the former section equilibrium points lying on the hyper-plane $Hyp = (x, y, z, r = 1)$ are associated with standard general relativity dynamics. The remaining points belonging in the bulk of the phase space Ψ are related with higher-dimensional brane effects.

The relevant equilibrium points of the autonomous system of equations (31) are summarized in table I, while the eigenvalues of the corresponding linearization (Jacobian) matrices are shown in table II. The empty universe (critical point E in table I) corresponds to decelerated expansion whenever $\omega_m > -2/3$. In the latter case it is always the past attractor for any trajectory in the phase space of the model. The matter-dominated solution (equilibrium point M) and the ultra-relativistic (scalar field-dominated) phase U^\pm , are always saddle critical points. The latter is associated with decelerating dynamics while the former M represents decelerating expansion whenever $\omega_m > -1/3$. Points M and U^\pm in table I correspond to the equilibrium points A and B of Ref. [14], respectively. Existence of the empty universe (critical point E in table I) is a distinctive feature of the higher-dimensional (brane) contributions.

More detailed information can be retrieved only after further simplification of the case to study. One way

to achieve further simplification is to study the ultra-relativistic regime where $\gamma = 0 \Rightarrow \gamma_L = \infty$. In the UR regime, thanks to the relationship

$$\gamma = 0 \Rightarrow y = \sqrt{3}x,$$

the autonomous system of ODE (31) simplifies down to a set of three ordinary differential equations:

$$\begin{aligned} y' &= -\frac{3\sqrt{3}\mu z^3}{2y} - \frac{3y}{2} - y\frac{H'}{H}, \\ z' &= \frac{\sqrt{3}\mu z^2}{2} - z\frac{H'}{H}, \quad r' = \frac{2r(r-1)}{2-r}\frac{H'}{H}, \\ \frac{H'}{H} &= -\frac{2-r}{2r}\{(\omega_m + 1)[3r - y^2 - 3z^2] + y^2\}. \end{aligned} \quad (34)$$

The reduced (three-dimensional) phase space in this simpler case is given by:

$$\Psi = \{(y, z, r) : 3(r-1) \leq y^2 + 3z^2 \leq 3r, \\ y \in \Re, z \geq 0, 0 < r \leq 1\}. \quad (35)$$

There are found five equilibrium points of the autonomous system of ODE (34) in Ψ . These points, together with their properties, are listed in table III, while the eigenvalues of the corresponding linearization matrices are shown in the table IV. By their overwhelming complexity, the eigenvalues of the linearization matrix corresponding to the fourth equilibrium point in table III (point SF) have not been included in table IV. It is

TABLE III: Properties of the equilibrium points of the autonomous system (34). Here $\gamma_m = \omega_m + 1$, while $\eta \equiv \mu(\sqrt{\mu^2 + 12} - \mu)$.

Equilibrium point	y	z	r	Existence	Ω_m	Ω_ϕ	ω_ϕ	q
E	0	0	0	Always	0	0	Undefined	$3\omega_m + 2$
M	0	0	1	"	1	0	Undefined	$\frac{3\omega_m + 1}{2}$
K	$\pm\sqrt{3}$	0	1	"	0	1	0	$\frac{1}{2}$
SF	$\pm\sqrt{\frac{\eta}{2}}$	$\frac{\eta}{(2\sqrt{3}\mu)}$	1	$\mu \geq 0$	0	1	$-1 + \frac{\eta}{6}$	$-1 - \frac{\eta}{4}$
MS	$\pm \frac{3\gamma_m}{\mu} \sqrt{-\frac{\gamma_m}{\omega_m}}$	$\frac{\sqrt{3}\gamma_m}{\mu}$	1	$\omega_m < 0$	$1 + \frac{3\gamma_m^2}{\mu^2\omega_m}$	$-\frac{3\gamma_m^2}{\mu^2\omega_m}$	ω_m	$\frac{3\omega_m + 1}{2}$

TABLE IV: Eigenvalues of the linearization matrices corresponding to the critical points in table III. The eigenvalues corresponding to the fourth point (SF) in Tab.III, have not been included due to their overwhelming complexity. Here $\xi \equiv \sqrt{\mu^2(9\omega_m^2 + 6\omega_m + 1) + 24(\omega_m^3 + 3\omega_m^2 + 3\omega_m + 1)}$.

Equilibrium point	λ_1	λ_2	λ_3
E	$3(\omega_m + 1)$	$3(\omega_m + 1)$	$3(2\omega_m + 1)/2$
M	$-3(\omega_m + 1)$	$3(\omega_m + 1)/2$	$3\omega_m/2$
K	$-3\omega_m$	-3	$3/2$
MS	$-3\gamma_m$	$\frac{3}{4}(1 - \omega_m + \xi/\mu)$	$-\frac{3}{4}(1 - \omega_m - \xi/\mu)$

worth recalling that the standard general relativity behaviour is associated with points in the phase space lying on the phase plane $Hyp = (y, z, r = 1)$. The remaining points belonging in the bulk of the phase space Ψ are associated with 5D effects.

Provided that $\omega_m > -1/2$ the empty universe – equilibrium point E in table III) – is always the past attractor in the phase space, *i.e.*, it represents the source critical point from which any phase path in Ψ originates. The matter-scaling solution (equilibrium point MS) is the late-time attractor provided that (the definition of the parameter ξ can be found in the caption of the table IV)

$$-(1 - \omega_m)\mu < \xi < (1 - \omega_m)\mu.$$

Otherwise, the scalar field-dominated solution SF is the late-time attractor.

We have to point out that the matter-scaling solution exists only if the equation of state (EOS) parameter of the background matter is a negative quantity: $\omega_m < 0$. This means that we can not have matter-scaling with background matter being dust ($\omega_m = 0$). Therefore, the usefulness of this equilibrium point to describe the current phase of the cosmic evolution is unclear. Unlike this, the

scalar field-dominated solution SF is always inflationary ($\eta \geq 0$ always) and could be associated with accelerated late-time cosmic dynamics. The matter-dominated solution M and the kinetic energy-dominated phase (equilibrium point K in table III), are always saddle points in the phase space.

It is worth noticing that the scalar field-dominated solution (point SF in table III) and the matter-scaling phase MS correspond to the points C and D in reference [14]. In figure 1 the trajectories in Ψ – the reduced phase space defined in (35) – originated by a given set of appropriated initial data, are drawn for the model of (31) in the ultra-relativistic approximation. The phase space graphics in figure 1 reveal the actual behavior of the RS dynamics: trajectories in phase space depart from the (singular) empty universe, possibly related with the initial (Big-Bang) singularity, and, at late times, approach to the plane $(y, z, 1)$, which is associated with standard four-dimensional behavior, in particular the trajectories approach to the SF point.

TABLE V: Properties of the equilibrium points of the autonomous system (36).

Equilibrium point	x	y	z	r	Existence	γ	Ω_m	Ω_ϕ	ω_ϕ	q
M	0	0	0	1	"	Undefined	1	0	Undefined	$(1 + 3\omega_m)/2$
U^\pm	1	$\pm\sqrt{3}$	0	1	"	0	0	1	0	1/2

TABLE VI: Eigenvalues of the linearization matrices corresponding to the critical points in table V.

Equilibrium point	λ_1	λ_2	λ_3	λ_4
M	$3\omega_m/2$	$3(1 + \omega_m)/2$	$3(1 + \omega_m)/2$	$3(1 + \omega_m)/2$
U^\pm	$-3\omega_m$	3	3/2	3/2

B. DBI-DGP Model

For an AdS throat and the quadratic self-interaction potential, the autonomous system of ODE (23)-(26), (16), (17), reduces down to the following four-dimensional autonomous system:

$$\begin{aligned}
x' &= -\frac{y^2}{2x} - \frac{\mu y z r}{2x^2}(z^2 - 2\gamma x^2) - x \frac{Q'}{Q}, \\
y' &= -6\gamma^2 \mu x z r - \frac{3(\gamma^2 + 1)}{2x}[\mu(z^2 - 2\gamma x^2)zr + \\
&\quad + xy] - y \frac{Q'}{Q}, \\
z' &= \frac{\mu y z^2 r}{2x} - z \frac{Q'}{Q}, \quad r' = r \left(\frac{1 - r^2}{1 + r^2} \right) \frac{Q'}{Q}. \quad (36)
\end{aligned}$$

The ratio Q'/Q is given by Eq. (27).

Recall that standard general relativity behavior is associated with points lying on the hyper-plane $HyP = (x, y, z, r = 1)$. The remaining points in the bulk of the phase space Ψ_\pm correspond to higher-dimensional behaviour.

The critical points of the autonomous system of ODE (36), together with their most important properties, are summarized in the table V. The eigenvalues of the linearization matrices corresponding to the critical points in table V are shown in table VI.

As for the DBI field trapped in a RS brane, in the present case the matter-dominated solution (M) and the UR phase (U^\pm) – dominated by the scalar field –, are found. In this case, however, these solutions show a quite different behaviour than the one found in the Randall-Sundrum case. Actually, the matter-dominated solution

is always a source point – the past attractor. The ultra-relativistic regime that is dominated by the scalar field mimics the cosmic evolution of a universe filled with dust. The above results are to be contracted with the results in the former subsection.

As before, a more detailed study of the asymptotic properties of the model (36) requires additional simplification. The ultra-relativistic approximation comes to our rescue. As long as one considers just large Lorentz boosts (amounting to vanishing γ) the relationship $y = \sqrt{3}x$ is verified. This relationship allows for further simplification of the autonomous system of ODE (36). Actually, in the UR regime the above system of equations can be simplified to the following three-dimensional autonomous system of ODE:

$$\begin{aligned}
y' &= -\frac{3}{2}y - \frac{3\sqrt{3}\mu z^3 r}{2y} - y \frac{Q'}{Q}, \\
z' &= \frac{\sqrt{3}}{2}\mu z^2 r - z \frac{Q'}{Q}, \\
r' &= \frac{r(1 - r^2)}{(1 + r^2)} \frac{Q'}{Q}. \quad (37)
\end{aligned}$$

The phase space for the autonomous system (37) can be defined in the following way. For the “+” - branch (the Minkowski cosmological phase):

$$\begin{aligned}
\Psi_+ &= \{(y, z, r) : 0 \leq y^2 + 3z^2 \leq 3, \\
&\quad y \in \mathfrak{R}, \quad z \geq 0, \quad r \in [1, \infty)\}, \quad (38)
\end{aligned}$$

while, for the self-accelerating “-” - branch, it is given by:

TABLE VII: Properties of the equilibrium points of the autonomous system (37). Here $\eta \equiv \mu(\sqrt{\mu^2 + 12} - \mu)$, while $\gamma_m \equiv \omega_m + 1$.

Equilibrium point	y	z	r	Existence	Ω_m	Ω_ϕ	ω_ϕ	q
M	0	0	1	"	1	0	Undefined	$\frac{3\omega_m+1}{2}$
K	$\pm\sqrt{3}$	0	1	"	0	1	0	$\frac{1}{2}$
SF	$\pm\sqrt{\frac{\eta}{2}}$	$\frac{\eta}{(2\sqrt{3}\mu)}$	1	$\mu \geq 0$	0	1	$-1 + \frac{\eta}{6}$	$-1 - \frac{\eta}{4}$
MS	$\pm \frac{3\gamma_m}{\mu} \sqrt{-\frac{\gamma_m}{\omega_m}}$	$\frac{\sqrt{3}\gamma_m}{\mu}$	1	$\omega_m < 0$	$1 + \frac{3\gamma_m^2}{\mu^2\omega_m}$	$-\frac{3\gamma_m^2}{\mu^2\omega_m}$	ω_m	$\frac{3\omega_m+1}{2}$

TABLE VIII: Eigenvalues of the linearization matrices corresponding to the first four critical points in table VII. Here $\Pi \equiv \sqrt{12\omega_m(6\omega_m + \mu\eta) + \mu^2(6 - \mu\eta)}$, while $\zeta \equiv \sqrt{24(1 + \omega_m)^3 + \mu^2(1 + 3\omega_m)^2}$.

Equilibrium point	λ_1	λ_2	λ_3
M	$3(\omega_m + 1)/2$	$3(\omega_m + 1)/2$	$3\omega_m/2$
K	$-3\omega_m$	$3/2$	$3/2$
SF	$\mu\eta/4$	$\frac{-12(2+\omega_m)+3\mu\eta}{8} + \frac{\sqrt{2}}{8}\Pi$	$\frac{-12(2+\omega_m)+3\mu\eta}{8} - \frac{\sqrt{2}}{8}\Pi$
MS	$\frac{3}{2}(1 + \omega_m)$	$\frac{3}{4}\left(\omega_m - 1 + \frac{\zeta}{\mu^2}\right)$	$\frac{3}{4}\left(\omega_m - 1 - \frac{\zeta}{\mu^2}\right)$

$$\Psi_- = \{(y, z, r) : 0 \leq y^2 + 3z^2 \leq 3, \\ y \in \mathbb{R}, z \geq 0, r \in]0, 1]\}. \quad (39)$$

There are four equilibrium points of the autonomous system of ODE (37). These critical points – together with their most salient features – are summarized in table VII.

If $\omega_m > 0$ then the matter-dominated solution (M) is the past attractor else the kinetic energy-dominated solution (K) is the past attractor in the phase space, as in [14], independent on which branch of the DGP is being considered (see figure 2, where we show the temporal evolution of the dynamical system) since, at early times, the brane effects can be safely ignored so that the standard cosmological dynamics is not modified.

The equilibrium point SF (scalar field-dominated solution) and MS (scaling dominated-solution) are always a saddle point. There is no future (late-time) attractor in the phase space of the model.

Phase trajectories in Ψ_+ (upper panels in figure 2) and in Ψ_- (lower panels in figure 2) originate from the source critical point M , corresponding to the standard 4D matter-dominated solution. Due to the non-linear nature of the DBI-type field the phase trajectories do not end at any given critical point.

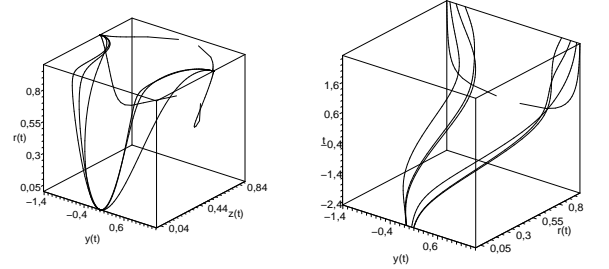


FIG. 1: Phase portrait generated by a given set of initial data ($\omega_m = 0$, $\mu = 2$) for the autonomous system of ODE (34), corresponding to the UR approximation of the DBI-RS model – left-hand panel, and the corresponding flux in time – right-hand panel. The values of the free parameter has been chosen so that the scalar field-dominated solution (point SF ($\pm 1.41, 0.58, 1$)) is the late-time attractor.

V. RESULTS AND DISCUSSION

The importance of the brane effects for the cosmic dynamics is well known. These effects can modify the gen-

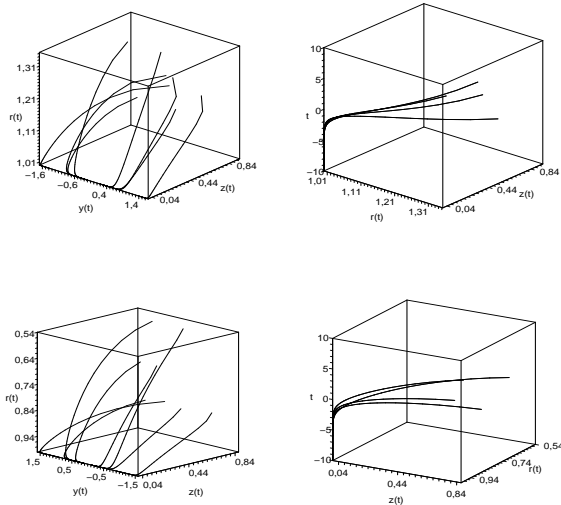


FIG. 2: Phase portrait for given initial data ($\omega_m = 0.1$, $\mu = 2$) for the autonomous system of ODE (37) corresponding to the UR approximation in the DBI-DGP model – left-hand panels, and the corresponding flux in time – right-hand panels. In the upper panels the Minkowski cosmological phase of the DGP is depicted, while in the lower panels the part of the phase space corresponding to the self-accelerating phase is shown. The past attractor is the matter-dominated solution (point M in table VII).

eral relativity laws at early times (UV modifications), as well as at late times (IR modifications). Actually, while the Randall-Sundrum brane model produces UV modifications of general relativity, in the Dvali-Gabadadze-Porrati braneworld IR modifications of the laws of gravity arise instead. In a similar way, the introduction of a non-linear Dirac-Born-Infeld type of field might modify the cosmic dynamics at early as well as at late times. This makes even more interesting the study of the combined effect of a DBI-type field trapped in the braneworld. Aim of the present paper has been, precisely, the study of the asymptotic properties of the latter kind of cosmological models.

In general case, the equilibrium points M y U in the phase space were found for the DBI-RS and DBI-DGP model (compare tables I and V). However the brane effects modify the stability properties of these points. While in the DBI-RS model the empty universe (equilibrium point E) is the past attractor, in the DBI-DGP model these point does not exist. The past attractor in the DBI-DGP model can be the matter-dominated so-

lution (equilibrium point M) or the kinetic dominated-solution (equilibrium point K). The critical points M and K are saddle point in the DBI-RS model. In both models the ultra-relativistic phase (point U^\pm) is also a saddle critical point in the corresponding phase spaces.

Because the past attractor in both models are different (equilibrium point E in DGP-RS model and equilibrium point M or U in the DBI-DGP model), it can be easily explained as due to the impact UV modifications produced by the Randall-Sundrum brane, have on the early-time cosmic dynamics; in the DBI-DGP case, at early times, the dynamics is general relativistic so that the stability properties of the matter-dominated phase in tables III and IV just coincide with the results of [14]. On the other hand, brane effects in the DBI-RS scenario dominate at early times modifying the nature of the equilibrium point M .

VI. CONCLUSION

In the present paper we aimed at studying the asymptotic properties of a DBI-type field trapped in the braneworld – for the AdS throat and the quadratic potential –, by means of the dynamical systems tools. The combined effect of the non-linear nature of the DBI field and of the higher-dimensional brane effects seem to produce a rich dynamics. Both brane contributions and non-linear DBI effects can modify the general relativity laws of gravity at late times as well as at early times. Here we focused in Randall-Sundrum and in Dvali-Gabadadze-Porrati braneworlds exclusively. In a sense this work can be considered as a natural completion of the one in Ref. [14] to consider the combined effect of the DBI-type field and of the braneworld.

We performed a thorough study of the phase space corresponding to the two scenarios. It is revealed that the matter-dominated solution, and the ultra-relativistic phase, are common to both of them. However, the stability properties of these points differ from one scenario to another. While in the DBI-RS model the empty (not necessarily inflationary) universe is always the past attractor, for the DBI-DGP scenario the past attractor is the matter or kinetic dominated-solution.

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